

Torsional Elastic Waves in Cylindrical Waveguide with Wedge Dislocation

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Abstract

We describe propagation of torsional elastic waves in cylindrical waveguide with wedge dislocation in the framework of geometric theory of defects. The defect changes the dispersion relation. For positive deficit angles, it increases the phase velocity, and decreases the group velocity of the wave.

1 Introduction

Ideal crystals are absent in nature, and most of their physical properties, such as plasticity, melting, growth, etc., are defined by defects of the crystalline structure. Therefore, a study of defects is a topical scientific question of importance for applications in the first place. At present, a fundamental theory of defects is absent in spite of the existence of dozens of monographs and thousands of articles.

One of the most promising approaches to the theory of defects is based on Riemann–Cartan geometry, which involves nontrivial metric and torsion. In this approach, a crystal is considered as a continuous elastic medium with a spin structure. If the displacement vector field is a smooth function, then there are only elastic stresses corresponding to diffeomorphisms of the Euclidean space. If the displacement vector field has discontinuities, then we are saying that there are defects in the elastic structure. Defects in the elastic structure are called dislocations and lead to the appearance of nontrivial geometry. Precisely, they correspond to a nonzero torsion tensor, equal to the surface density of the Burgers vector. Defects in the spin structure are called disclinations. They correspond to nonzero curvature tensor, curvature tensor being the surface density of the Frank vector.

The idea to relate torsion to dislocations appeared in the 1950s [1–4]. This approach is still being successfully developed (note reviews [5–11]), and is often called the gauge theory of dislocations.

Some time ago we proposed the geometrical theory of defects [12–14]. Our approach is essentially different from others in two respects. Firstly, we do not have the displacement vector field and rotational vector field as independent variables because, in general, they are not continuous. Instead, the triad field and $\mathbb{SO}(3)$ -connection are considered as

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independent variables. If defects are absent, then the triad and $\mathbb{SO}(3)$ -connection reduce to partial derivatives of the displacement and rotational angle vector fields. In this case the latter can be reconstructed. Secondly, the set of equilibrium equations is different. We proposed purely geometric set which coincides with that of Euclidean three dimensional gravity with torsion. The nonlinear elasticity equations and principal chiral $\mathbb{SO}(3)$ model for the spin structure enter the model through the elastic and Lorentz gauge conditions [14–16] which allow to reconstruct the displacement and rotational angle vector fields in the absence of dislocations in full agreement with classical models.

The advantage of the geometric theory of defects is that it allows one to describe single defects as well as their continuous distributions.

In the present paper, we consider propagation of torsional elastic waves in cylindrical waveguide with wedge dislocation. It is a classical problem which was solved in the absence of defect long ago within the elasticity theory (see, for example, [17]). We show that the wedge dislocation changes the dispersion relation. For positive deficit angles the phase velocity increases and group velocity decreases. Negative deficit angles have opposite consequences.

2 Elastic waves in media with dislocations

Let us introduce notation and remind necessary facts from differential geometry. We consider elastic media as topologically trivial manifold which is diffeomorphic to Euclidean space \mathbb{R}^3 . Denote Cartesian coordinate system by x^i , $i = 1, 2, 3$. Without dislocations, the elastic media is described by the Euclidean metric δ_{ij} . If dislocations are present then the metric becomes nontrivial:

$$\delta_{ij} \mapsto g_{\mu\nu}(x) = e_\mu^i e_\nu^j \delta_{ij}, \quad \mu, \nu = 1, 2, 3,$$

where $e_\mu^i(x)$ is the triad. The triad field defines the orthonormal basis of the tangent spaces at each point $e_i := e^\mu_i \partial_\mu$, where e^μ_i is the inverse triad. We shall see that this basis is more convenient in applications.

Assume that relative displacements for elastic deformations are much smaller then deformations caused by defects:

$$\partial_\mu u^i \ll e_\mu^i. \quad (1)$$

where $u^i(t, x)$ are components of the displacement vector field with respect to orthonormal basis in the tangent space e_i . Then in the first approximation, the elastic waves propagate in Riemannian space with nontrivial metric produced by dislocations. Here we neglect changes in the metric produced by elastic waves themselves. Therefore for elastic waves in media with defects we postulate the following wave equation

$$\rho_0 \ddot{u}^i - \mu \tilde{\Delta} u^i - (\lambda + \mu) \tilde{\nabla}^i \tilde{\nabla}_j u^j = 0, \quad (2)$$

where λ and μ are Lamé coefficients, $\tilde{\Delta} := \tilde{\nabla}^i \tilde{\nabla}_i$ is the covariant Laplace–Beltrami operator for the triad field e_μ^i , and $\tilde{\nabla}_i$ is the covariant derivative. The explicit form of the covariant derivative is

$$\tilde{\nabla}_i u^j := e^\mu_i \tilde{\nabla}_\mu u^j = e^\mu_i (\partial_\mu u^j + u^k \tilde{\omega}_{\mu k}^j), \quad (3)$$

where $\tilde{\omega}_{\mu k}^j$ is the $\mathbb{SO}(3)$ -connection constructed for zero torsion.

Note that the displacement vector field u^i is not the total displacement vector field of points in the media with dislocations. In those regions where defects are absent, the total

displacement vector is $u_D^i + u^i$ where u_D^i is the dislocation part of the displacement vector field which produces the triad $e_\mu^i := \partial_\mu u_D^i$. Note also that the smallness of relative elastic deformations (1) is meaningful at those regions where displacements u_D^i are not defined (they are not continuous functions if dislocations are present).

The expression for the deformation tensor σ_{ij} is needed to pose the boundary value problem for wave equation (2). In the presence of defects the deformation tensor is defined as follows

$$\epsilon_{ij} := \frac{1}{2}(e^\mu_i \tilde{\nabla}_\mu u_j + e^\mu_j \tilde{\nabla}_\mu u_i). \quad (4)$$

Thus the existence of dislocations in media results in the introduction of the triad field $e_\mu^i(x)$, metric $g_{\mu\nu}(x)$, and replacement of partial derivatives by covariant ones.

Remember that lowering of Latin indices is performed using the Euclidean metric, $u_i := \delta_{ij} u^j$, and it commutes with covariant differentiation.

2.1 Torsional waves in cylindrical waveguide with wedge dislocation

We consider cylindrical waveguide of radius a with a wedge dislocation. Its axis is assumed to coincide with the axis of cylinder. We choose cylindrical coordinate system r, φ, z , the z axis coinciding with the cylinder axis as well. Let the wedge dislocation has deficit angle $2\pi\theta$ (see Fig.1) If $-1 < \theta < 0$, then the wedge is cut out from the cylinder. Dislocation

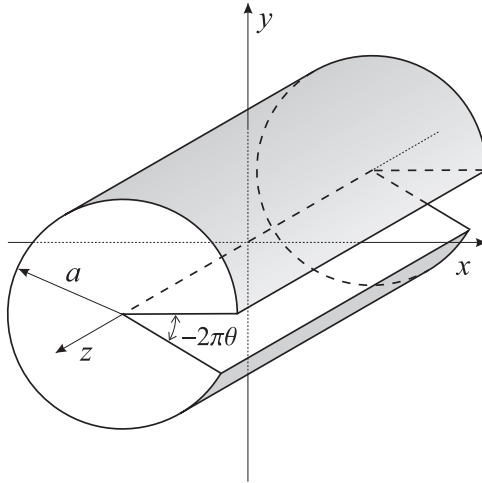


Figure 1: Wedge dislocation with the deficit angle $2\pi\theta$. For negative and positive θ , the wedge is cut out or added, respectively.

is absent for $\theta = 0$. For positive deficit angle, $\theta > 0$, the wedge of media is added to the cylinder.

The cylinder contains wedge dislocation, and it creates nontrivial metrics. The metric becomes noneuclidean [15]

$$ds^2 = \left(\frac{r}{a}\right)^{2\gamma-2} \left(dr^2 + \frac{\alpha^2 r^2}{\gamma^2} d\varphi^2\right) + dz^2, \quad (5)$$

where $\alpha := 1 + \theta$, and we introduce dimensionless constant

$$\gamma := -\theta b + \sqrt{\theta^2 b^2 + 1 + \theta}. \quad (6)$$

The constant

$$b := \frac{\sigma}{2(1 - \sigma)}$$

is defined by dimensionless Poisson ratio

$$\sigma := \frac{\lambda}{2(\lambda + \mu)},$$

which characterize elastic properties of media. Thermodynamical arguments restrict possible values of Poisson ratio to $-1 \leq \sigma \leq 1/2$ [18].

The limit

$$\theta \rightarrow 0, \quad \alpha \rightarrow 1, \quad \gamma \rightarrow 1 \quad (7)$$

corresponds to absence of the dislocation.

Metric (5) is rotationally and translationally invariant. Two dimensional metric on sections $z = \text{const}$ is the metric of conical singularity which arises due to the wedge dislocation. It is important that this metric is written in elastic gauge and therefore is consistent with the elasticity theory [15].

Metric for the wedge dislocation (5) is defined for all deficit angles $\theta > -1$ and all values of radial coordinate $0 < r < a$. It is degenerate on the dislocation axis $r = 0$. Metric (5) coincide with the induced metric in classical elasticity theory only for small deficit angles and near the boundary of the cylinder $r \sim a$ [19, 15].

Metric for the wedge dislocation defines the triad field which we choose to be diagonal:

$$e_r^{\hat{r}} = \left(\frac{r}{a}\right)^{\gamma-1}, \quad e_\varphi^{\hat{\varphi}} = \left(\frac{r}{a}\right)^{\gamma-1} \frac{\alpha r}{\gamma}, \quad e_z^{\hat{z}} = 1, \quad (8)$$

where $\mu = r, \varphi, z$ and $i = \hat{r}, \hat{\varphi}, \hat{z}$. In the absence of dislocation, $\gamma = 1, \alpha = 1$, it defines the usual orthonormal basis for tangent spaces in cylindrical coordinates. The inverse triad is

$$e_r^{\hat{r}} = \left(\frac{a}{r}\right)^{\gamma-1}, \quad e_\varphi^{\hat{\varphi}} = \left(\frac{a}{r}\right)^{\gamma-1} \frac{\gamma}{\alpha r}, \quad e_z^{\hat{z}} = 1. \quad (9)$$

To find the explicit form of the wave operator, we need to compute Christoffel's symbols and components of $\mathbb{SO}(3)$ -connection. Straightforward calculations show that only for Christoffel's symbols differ from zero:

$$\begin{aligned} \tilde{\Gamma}_{rr}^r &= \frac{\gamma - 1}{r}, \\ \tilde{\Gamma}_{r\varphi}^\varphi &= \tilde{\Gamma}_{\varphi r}^\varphi = \frac{\gamma}{r}, \\ \tilde{\Gamma}_{\varphi\varphi}^r &= -\frac{\alpha^2 r}{\gamma}. \end{aligned} \quad (10)$$

Triad (8) defines also components of $\mathbb{SO}(3)$ -connection

$$\omega_{\mu i}^j = -\partial_\mu e_\nu^j e^\nu_i + e^\nu_i \tilde{\Gamma}_{\mu\nu}^\rho e_\rho^j.$$

Only two components differ from zero:

$$\omega_{\varphi\hat{r}}^{\hat{\varphi}} = -\omega_{\varphi\hat{\varphi}}^{\hat{r}} = \alpha. \quad (11)$$

Now we can compute the Laplacian

$$\begin{aligned} \tilde{\Delta} u_i &= g^{\mu\nu} \partial_{\mu\nu}^2 u_i - g^{\mu\nu} \partial_\mu \omega_{\nu i}^j u_j - 2g^{\mu\nu} \omega_{\mu i}^j \partial_\nu u_j - \\ &\quad - g^{\mu\nu} \tilde{\Gamma}_{\mu\nu}^\rho (\partial_\rho u_i - \omega_{\rho i}^j u_j) + g^{\mu\nu} \omega_{\mu i}^k \omega_{\nu k}^j u_j. \end{aligned} \quad (12)$$

Substitution of explicit expressions for Christoffel's symbols (10) and $\mathbb{SO}(3)$ -connection (11) yields the following expressions:

$$\begin{aligned}
\tilde{\Delta}u_{\hat{r}} &= \left(\frac{a}{r}\right)^{2\gamma-2} \frac{1}{r} \partial_r (r \partial_r u_{\hat{r}}) + \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{\alpha^2 r^2} \partial_{\varphi\varphi}^2 u_{\hat{r}} + \partial_{zz}^2 u_{\hat{r}} - \\
&\quad - \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{r^2} u_{\hat{r}} - 2 \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{\alpha r^2} \partial_{\varphi} u_{\hat{\varphi}}, \\
\tilde{\Delta}u_{\hat{\varphi}} &= \left(\frac{a}{r}\right)^{2\gamma-2} \frac{1}{r} \partial_r (r \partial_r u_{\hat{\varphi}}) + \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{\alpha^2 r^2} \partial_{\varphi\varphi}^2 u_{\hat{\varphi}} + \partial_{zz}^2 u_{\hat{\varphi}} - \\
&\quad - \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{r^2} u_{\hat{\varphi}} + 2 \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{\alpha r^2} \partial_{\varphi} u_{\hat{r}}, \\
\tilde{\Delta}u_{\hat{z}} &= \left(\frac{a}{r}\right)^{2\gamma-2} \frac{1}{r} \partial_r (r \partial_r u_{\hat{z}}) + \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{\alpha^2 r^2} \partial_{\varphi\varphi}^2 u_{\hat{z}} + \partial_{zz}^2 u_{\hat{z}}.
\end{aligned} \tag{13}$$

Corresponding wave equations (2) have many solutions. The simplest one describes torsional waves for which only the angular component $u_{\hat{\varphi}}$ differs from zero and does not depend on the angular coordinate φ :

$$u_{\hat{r}} = 0, \quad u_{\hat{\varphi}} = u_{\hat{\varphi}}(t, r, z), \quad u_{\hat{z}} = 0.$$

It is easy to see that torsional waves take place without media compression

$$\epsilon := \tilde{\nabla}_i u^i = 0.$$

For these oscillations, wave equations $\tilde{\square}u_{\hat{r}} = 0$ and $\tilde{\square}u_{\hat{z}} = 0$ are identically satisfied, and we are left with one wave equation

$$\frac{1}{c_T^2} \ddot{u}_{\hat{\varphi}} - \left(\frac{a}{r}\right)^{2\gamma-2} \frac{1}{r} \partial_r (r \partial_r u_{\hat{\varphi}}) - \partial_{zz}^2 u_{\hat{\varphi}} + \left(\frac{a}{r}\right)^{2\gamma-2} \frac{\gamma^2}{r^2} u_{\hat{\varphi}} = 0. \tag{14}$$

We look for solution of this equation in the plain wave form

$$u_{\hat{\varphi}} = \text{re} \left(U e^{i(kz - \omega t)} \right),$$

where $U(r) \in \mathbb{R}$ is the wave amplitude, $k \in \mathbb{R}$ is the wave vector, and $\omega \in \mathbb{R}$ is the wave frequency. Then wave equation (14) takes the form

$$r \partial_r (r \partial_r U) + \kappa^2 a^2 \left(\frac{r}{a}\right)^{2\gamma} U - \gamma^2 U = 0, \tag{15}$$

where

$$\kappa^2 := \frac{\omega^2}{c_T^2} - k^2. \tag{16}$$

Now we introduce new radial coordinate

$$r = a r'^{\frac{1}{\gamma}}, \quad 0 < r' < 1.$$

Then equation (16) reduces to the Bessel equation

$$r'^2 \frac{d^2 U}{dr'^2} + r' \frac{dU}{dr'} + \lambda^2 r'^2 U - U = 0, \tag{17}$$

where

$$\lambda^2 := \frac{\kappa^2 a^2}{\gamma^2}.$$

A general solution to the Bessel equation has two integration constants. We require it to be finite at $r = 0$. Then the amplitude takes the form

$$U = AJ_1 \left(\lambda \left(\frac{r}{a} \right)^\gamma \right), \quad (18)$$

where $A \in \mathbb{R}$ is the integration constant and J_1 is the Bessel function of the first kind of order one (see, for example, [20]).

If dislocation is absent, $\gamma = 1$, then the solution takes the well known form (see, for example, [17])

$$U \xrightarrow{\gamma \rightarrow 1} AJ_1(\kappa r)$$

To impose boundary conditions on the cylinder surface, we need explicit form of deformation tensor (4). Straightforward calculations yeild the following expressions:

$$\begin{aligned} \epsilon_{\hat{r}\hat{r}} &= \left(\frac{a}{r} \right)^{\gamma-1} \partial_r u_{\hat{r}}, & \epsilon_{\hat{r}\hat{\varphi}} &= \frac{1}{2} \left(\frac{a}{r} \right)^{\gamma-1} \left[\partial_r u_{\hat{\varphi}} + \frac{\gamma}{\alpha r} \partial_\varphi u_{\hat{r}} - \frac{\gamma}{r} u_{\hat{\varphi}} \right], \\ \epsilon_{\hat{\varphi}\hat{\varphi}} &= \left(\frac{a}{r} \right)^{\gamma-1} \frac{\gamma}{\alpha r} (\partial_\varphi u_{\hat{\varphi}} + \alpha u_{\hat{r}}), & \epsilon_{\hat{r}\hat{z}} &= \frac{1}{2} \left[\left(\frac{a}{r} \right)^{\gamma-1} \partial_r u_{\hat{z}} + \partial_z u_{\hat{r}} \right], \\ \epsilon_{\hat{z}\hat{z}} &= \partial_z u_{\hat{z}}, & \epsilon_{\hat{\varphi}\hat{z}} &= \frac{1}{2} \left[\left(\frac{a}{r} \right)^{\gamma-1} \frac{\gamma}{\alpha r} \partial_\varphi u_{\hat{z}} + \partial_z u_{\hat{\varphi}} \right]. \end{aligned} \quad (19)$$

Only two components

$$\epsilon_{\hat{r}\hat{\varphi}} = \epsilon_{\hat{\varphi}\hat{r}} = \frac{1}{2} \left(\frac{a}{r} \right)^{\gamma-1} \left[\partial_r u_{\hat{\varphi}} - \frac{\gamma}{r} u_{\hat{\varphi}} \right]$$

differ from zero for torsional waves. We require the cylinder surface to be free. Then the elastic forces on the boundary must be zero. It yeilds the boundary condition

$$\left[\partial_r u_{\hat{\varphi}} - \frac{\gamma}{r} u_{\hat{\varphi}} \right]_{r=a} = 0. \quad (20)$$

For solution (18), it takes the form

$$\lambda J_1'(\lambda) - J_1(\lambda) = 0,$$

where prime denotes differentiation of the Bessel function with respect to its argument. There is equality

$$J_1'(\lambda) = J_0(\lambda) - \frac{1}{\lambda} J_1(\lambda).$$

Then the boundary condition is equivalent to the equality which defines the dispersion relation

$$\frac{\kappa a}{\gamma} = \xi \quad \Leftrightarrow \quad \omega = c_T \sqrt{k^2 + \frac{\gamma^2 \xi^2}{a^2}}. \quad (21)$$

where ξ is a root of the equation

$$\xi J_0(\xi) = 2J_1(\xi). \quad (22)$$

Bessel functions J_ν have the following asymptotics for large argument, $\xi \gg 1$, $\xi \gg \nu$,

$$\begin{aligned} J_0(\lambda) &\approx \sqrt{\frac{2}{\pi\lambda}} \cos\left(\lambda - \frac{\pi}{4}\right), \\ J_1(\lambda) &\approx \sqrt{\frac{2}{\pi\lambda}} \sin\left(\lambda - \frac{\pi}{4}\right). \end{aligned}$$

Then equation (21) takes the form

$$\xi \cos\left(\xi - \frac{\pi}{4}\right) = 2 \sin\left(\xi - \frac{\pi}{4}\right)$$

for large argument. This equation has countable set of roots, each of them defining the dispersion relation.

The phase velocity of torsional waves $v := \omega/k$ has the form

$$v = c_T \sqrt{1 + \frac{\gamma^2 \xi^2}{a^2 k^2}}. \quad (23)$$

It is not difficult to calculate the group velocity

$$v_g := \frac{d\omega}{dk} = \frac{c_T^2}{v}. \quad (24)$$

If dislocation is absent, equation (21) for ξ remains the same. Relation (24) between phase and group velocities preserves its form. Thus the presence of the wedge dislocation changes only dispersion relation (21).

For positive deficit angles $\gamma > 1$, the phase velocity increases and the group velocity decreases as the consequence of equations (23) and (24).

For small deficit angles the first correction has the form

$$\gamma \approx 1 + \theta \frac{1 - 2\sigma}{2(1 - \sigma)}$$

and

$$\omega \approx c_T \sqrt{k^2 + \frac{\xi^2}{a^2}} \left[1 + \theta \frac{1 - 2\sigma}{2(1 - \sigma)} \frac{1}{1 + \frac{k^2 a^2}{\xi^2}} \right]. \quad (25)$$

It is linear in deficit angle of the dislocation.

3 Conclusion

We showed that the presence of the wedge dislocation in cylindrical waveguide leads to changing of the dispersion relation for torsional elastic waves. The difference increases as long as the deficit angle increases. For positive deficit angle, the phase velocity of the wave increases and group velocity decreases. For negative deficit angle, the behavior is opposite.

Solution of this problem within the classical elasticity encounters essential technical problems. Indeed, relative displacements are not small in the neighbourhood of the dislocation axis even for small deficit angles, and the expression for the induced metric is much more complicated [19, 15]. Solution of this problem within the geometric theory of defects as simpler.

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